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Operator content of the cyclic solid-on-solid models

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Abstract. The operator content of the cyclic solid-on-solid models is derived under general boundary conditions by mapping the row-to-row transfer matrix onto the six-vertex model and XXZ Heisenberg chain. For the shift boundary conditions, the dimensions and spins of the primary operators are derived analytically from finite-size corrections to the scaling spectra. The full operator content agrees with results for a quantum chain model having the same local symmetry.

1. Introduction

Recently many hierarchies of solvable two-dimensional lattice models have been found (Andrews *et al* 1984, Pasquier 1987a, b, c, Kuniba and Yajima 1988a, b). The adjacency graphs of these models are identical to the Dynkin diagrams of classical Lie algebras and their affine extensions. One such model is the cyclic solid-on-solid (CSOS) model associated with the $A_{L-1}^{(1)}$ algebra (Pearce and Seaton 1988, 1989, Kuniba and Yajima 1988b). The adjacency diagram for the CSOS model is shown in figure 1. The model is solvable on a three-dimensional manifold in the full $(4L-1)$-dimensional thermodynamic space and its free energy and local height probabilities have been obtained.

The partition function of a conformal invariant critical lattice model on a torus is, apart from a non-universal bulk term, modular invariant and universal (Cardy 1986a, b,

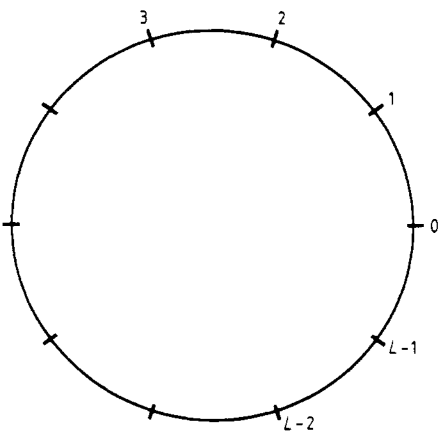

Figure 1. The adjacency diagram of the L-state CSOS model.

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Itzykson and Zuber 1986, Di Francesco *et al* 1987). It can be constructed from the operator content of the model which describes the universal eigenvalue spectra of the row-to-row transfer matrix (Cardy 1986, Rittenberg 1988). The operator content is sensitive to the boundary conditions (BC) and it is of interest to derive the full operator content of critical models under general BC.

At criticality, the exact solution manifold of the CSOS model collapses onto a line parametrised by a single spectral parameter u. In this case the Boltzmann weights of a face become identical to those of the six-vertex model. In this paper we use this connection with the six-vertex model to derive the full operator content of the CSOS model under general BC compatible with toroidal boundary conditions. We do not consider free or fixed boundary conditions.

In the next section we define the CSOS models and introduce $2L$ BC as elements of the local symmetry group. These consist of L shift and L reflection BC. In § 3 the transfer matrix of the critical CSOS model with shift BC is mapped to that of the six-vertex model with a seam along the first column. We use the Bethe ansatz and associated analytic methods (Woynarowich 1987, Hamer *et al* 1987) to derive the dimensions and spins of the primary operators from finite-size effects (Cardy 1986, 1989). In § 4, we consider the reflection BC. In this case, the transfer matrix of the CSOS model corresponds to the six-vertex model with antiperiodic BC on horizontal arrows. This maps, in the extreme anisotropic limit, to the XXZ Heisenberg chain with the charge conjugation BC. We are therefore able to use the results for the XXZ chain (Alcaraz *et al* 1988a) to deduce the full operator content. In the last section, we discuss our results and compare with the results of von Gehlen *et al* (1988) for quantum chain models having the same symmetry. For L odd, we are led to introduce an extended version of the model for a proper interpretation of the spectra.

2. The model, symmetry and boundary conditions

The CSOS model is an L-state IRF (or interaction-round-a-face) model on the square lattice (Baxter 1982). The spins or heights on each lattice site take the integer values $0, 1, \ldots, L-1$ and are denoted by a, b, c, d, etc; all heights being interpreted modulo L. The heights of adjacent sites are restricted to differ by ± 1 mod L and the height L is identified with 0. In the CSOS model there are thus six types of allowed non-zero face weights. To each type one can assign a vertex configuration as shown in figure 2. Strictly speaking, when $L=4$ two additional vertices, with all arrows going in or out, satisfy the adjacency conditions. These occur in the eight-vertex model but are not allowed in the CSOS models.

The elliptic function parametrisation of the CSOS face weights is given in Pearce and Seaton (1988, 1989). It is

$$\begin{aligned} W_1 = W_2 &= \frac{\theta_1(\lambda - u)}{\theta_1(\lambda)} \\ W_3 = W_4 &= \frac{\theta_1(u)}{\theta_1(\lambda)} \frac{[\theta_4(w_{a-1})\theta_4(w_{a+1})]^{1/2}}{\theta_4(w_a)} \\ W_5 &= \frac{\theta_4(w_a + u)}{\theta_4(w_a)} \end{aligned} \tag{1}$$

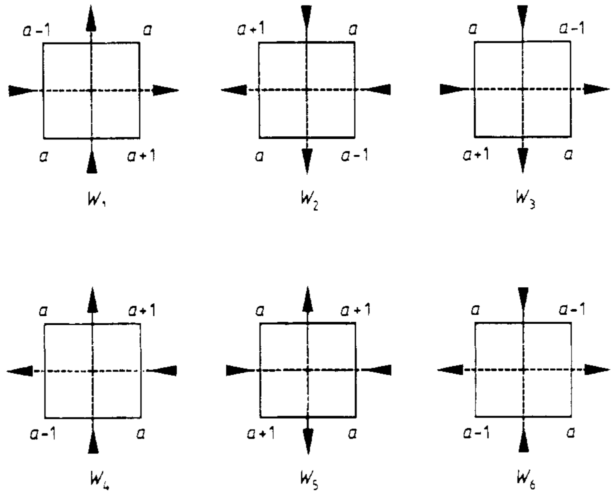

Figure 2. The six types of non-vanishing face weights of the CSOS model and assignment of six-vertex arrow configurations.

$$W_6 = \frac{\theta_4(w_a - u)}{\theta_4(w_a)}$$

where θ_1 and θ_4 are the usual elliptic functions (Gradshteyn and Ryzhik 1980) and $w_a = w_0 + a\lambda$. The crossing parameter λ can take any of a set of discrete values:

$$\lambda = \pi s / L \tag{2}$$

with $s = 1, 2, \ldots, L-1$ coprime to L. L and s are the model parameters and w_0, u and the nome of the elliptic functions are the thermodynamic variables. In general, the face weights contain an angle variable w_0 and are height dependent. However, at criticality, these face weights become invariant under a shift of heights and with a suitable normalisation assume the simple form

$$\begin{aligned} &W_1 = W_2 = 1 \\ &W_3 = W_4 = \sin u / \sin(\lambda - u) \\ &W_5 = W_6 = \sin \lambda / \sin(\lambda - u). \end{aligned} \tag{3}$$

These are precisely the weights of the critical six-vertex model.

To discuss symmetry and BC, let us introduce $2L$ operators acting on the heights by

$$\begin{aligned} &\Sigma^l a = l + a \qquad \text{(shifts)} \\ &\Sigma^l C a = l - a \qquad \text{(reflections)} \\ &l \in S_L \equiv \{0, 1, \ldots, L-1\}. \end{aligned} \tag{4}$$

These operators, which leave the face weights (3) invariant, form the dihedral group D_L, which is the local symmetry group of the model. To each $g \in \mathrm{D}_L$, we associate a BC $a_{N+1} = g a_1$, where N is the width of the lattice. When L is even, $N + l$ must be even for both Σ^l and $\Sigma^l C$. The transfer matrix V_g, associated with the BC g, is then defined via its elements as

$$V_g(\boldsymbol{a} \,|\, \boldsymbol{b}) = \prod_{i=1}^{N} W(a_i, a_{i+1}, b_{i+1}, b_i) \tag{5}$$

where $\boldsymbol{a}=\{a_1,\ldots,a_N\}$ and $\boldsymbol{b}=\{b_1,\ldots,b_N\}$ are arbitrary allowed configurations of a row, $a_{N+1}=ga_1$, $b_{N+1}=gb_1$ and $W(a,b,c,d)$ is the weight of a face with heights a, b, c, d given counterclockwise starting from the lower left corner. A row configuration $\boldsymbol{a}=\{a_1,\ldots,a_N\}$ can also be represented by $(a_1,\boldsymbol{\sigma})$ where $\boldsymbol{\sigma}=\{\sigma_1,\ldots,\sigma_N\}$ and $\sigma_i=a_{i+1}-a_i$. With the identification of $\sigma_i=1$ (-1) to the up (down) arrow in the ith vertical bond, $\boldsymbol{\sigma}$ also stands for an arrow configuration. The global symmetry group G_g of $\boldsymbol{V}_g$ consists of all elements of D_L which commute with g. It is D_L for $g=\Sigma^0$ and also for $\Sigma^{L/2}$ when L is even but Z_L for other Σ^l. Similarly, it is $Z_2\times Z_2$ (Z_2) for Σ^lC, $l\in S_L$ when L is even (odd). Group elements g in the same conjugacy class lead to the same $\boldsymbol{V}_g$ and hence to the same operator content. The conjugacy classes of D_L are

$$\{\Sigma^0\},\{\Sigma^l,\Sigma^{L-l}\}\qquad(l=1,\ldots,(L-1)/2);\qquad\{\Sigma^lC\mid l\in S_L\}\tag{6}$$

for odd L and

$$\begin{aligned}&\{\Sigma^0\},\{\Sigma^{L/2}\},\{\Sigma^l,\Sigma^{L-l}\}\qquad(l=1,\ldots,L/2-1);\\&\{\Sigma^lC\mid l=\text{even}\},\{\Sigma^lC\mid l=\text{odd}\}\end{aligned}\tag{7}$$

for even L.

3. Operator content for $g=\Sigma^l$

Consider the row-to-row transfer matrix $\boldsymbol{V}_g$ under the BC $g=\Sigma^l$, $l\in S_L$. An allowed row configuration $(a_1,\boldsymbol{\sigma})$ then satisfies the BC only if

$$\sum_{i=1}^{N}\sigma_i\equiv 2Q=l+nL\tag{8}$$

where the winding number n takes the values

$$\begin{aligned}&n\in\boldsymbol{Z}&&\text{for } N+l \text{ even and } L \text{ even}\\&n\in 2\boldsymbol{Z}&&\text{for } N+l \text{ even and } L \text{ odd}\\&n\in 2\boldsymbol{Z}+1&&\text{for } N+l \text{ odd and } L \text{ odd}\end{aligned}\tag{9}$$

with $\boldsymbol{Z}=\{0,\pm1,\pm2,\ldots\}$. The 'charge' Q defined in (8) is a conserved quantity because of the six-vertex constraints, and hence each charge sector can be treated separately. Furthermore, exploiting the Z_L symmetry, we can separate $\boldsymbol{V}_g$ into L sectors of 'Z_L charge' $P\in S_L$ via a similarity transformation. Hence

$$\boldsymbol{V}_g=\bigoplus_{P\in S_L}\boldsymbol{V}_g^{(P)}\tag{10}$$

where $\oplus$ denotes direct sum and the elements of $\boldsymbol{V}_g^{(P)}$ are given by

$$V_g^{(P)}(\boldsymbol{\sigma}\mid\boldsymbol{\sigma}')=\omega^{-P}V_g(0,\boldsymbol{\sigma}\mid-1,\boldsymbol{\sigma}')+\omega^{P}V_g(0,\boldsymbol{\sigma}\mid 1,\boldsymbol{\sigma}')\tag{11}$$

with $\omega=\exp(2\pi\mathrm{i}/L)$. Apart from the $\omega^{\pm P}$ factors, the right-hand side of (11) is precisely the transfer matrix elements of the six-vertex model, the two terms corresponding to the two horizontal arrow directions. The extra phase factors can be incorporated simply by introducing a seam along the first column of the lattice where the vertex weights are modified to

$$\begin{aligned} W_i' &= \omega^{-P} W_i \qquad \text{for } i = 1, 3, 5 \\ W_i' &= \omega^{P} W_i \qquad \text{for } i = 2, 4, 6 \end{aligned} \tag{12}$$

where W_i are given by (3).

The transfer matrix of the csos model with BC Σ^l is thus a direct sum over appropriate charge sectors of those of the six-vertex model with a seam. Fortunately, the seam does not prevent diagonalisation using the Bethe ansatz. A straightforward calculation following Baxter (1982) yields the eigenvalue expressions for $V_g^{(P)}$ in the sector Q as

$$\Lambda = \omega^{-P} \prod_{j=1}^{M} L(x_j, u) + \omega^{P} \left(\frac{\sin u}{\sin(\lambda - u)} \right)^N \prod_{j=1}^{M} L(-x_j, \lambda - u) \tag{13}$$

where $M = N/2 - Q$ is the number of down arrows and

$$L(x, u) = \frac{\exp(2u\mathrm{i}) - \exp(2x - \lambda\mathrm{i})}{\exp(2x) - \exp(2u\mathrm{i} - \lambda\mathrm{i})}. \tag{14}$$

The zeros $\{x_j \mid j = 1, 2, \ldots, M\}$ are the solutions of the Bethe ansatz equations:

$$I_j/N = Z_N(x_j) \qquad j = 1, 2, \ldots, M \tag{15}$$

$$Z_N(x) = (2\pi)^{-1} \left(\Theta(x, \lambda/2) - N^{-1} \sum_{j=1}^{M} \Theta(x - x_j, \lambda) \right) - \phi/2\pi N \tag{16}$$

where $\phi = 4\pi P/L$ and

$$\Theta(x, \lambda) = 2\tan^{-1}(\tanh x / \tan \lambda). \tag{17}$$

For the largest eigenvalue in each sector the integers or half-integers I_j should be chosen as

$$I_j = -\frac{M+1}{2} + j \qquad (j = 1, \ldots, M) \tag{18}$$

for $\phi/2\pi < \frac{1}{2}$. Other leading eigenvalues are obtained by choosing $I_j = -(M+1)/2 + j + m$ $(m \in Z)$ (Alcaraz *et al* 1988b). This is the same as using (18) and shifting I_j in (15) to $I_j + m$. We absorb this shift into the definition of $Z_N(x)$ and redefine ϕ as

$$\phi/2\pi = 2P/L + m. \tag{19}$$

By varying m, we obtain a sequence of eigenvalues. These eigenvalues Λ depend on l and n through Q in (8) and on P and m through ϕ.

The Bethe ansatz equations (15)-(18) are the same as those obtained for the XXZ chain under a twisted BC with twisting angle ϕ (Alcaraz *et al* 1988b, Hamer *et al* 1987). Moreover, in the quantum chain limit $u \to 0$, the eigenvalue expression becomes

$$\ln \Lambda = -2\pi P\mathrm{i}/L + \mathrm{i} \sum_{j=1}^{M} \Theta(x_j, \lambda/2) + \frac{2u}{\sin \lambda} \sum_{j=1}^{M} \{\cos[\Theta(x_j, \lambda/2)] + \cos \lambda\}. \tag{20}$$

The real part of the right-hand side of (20) is the XXZ chain energy. Therefore the operator content can be directly inferred, at least in the $u \to 0$ limit, from known results for the XXZ chain (Alcaraz *et al* 1988a, Rittenberg 1988). One can go further and calculate analytically the finite-size corrections to the eigenvalues following the methods of Woynarovich and Eckle (1987), Woynarovich (1987), Hamer *et al* (1987) and Hamer and Batchelor (1988). In this way the scaling dimensions and spins of the corresponding scaling oeprators can be obtained (Cardy 1986a, b).

The difference between the *XXZ* chain, treated by the above authors, and the CSOS model is in the expression for the eigenvalues. For simplicity, we first treat the case $0<u<\lambda/2$. The result then readily extends to $0<u<\lambda$ by symmetry and continuity arguments. When $0<u<\lambda/2$, the first term in (13) dominates the second term exponentially, so we drop the second term. The eigenvalues for finite N can then be written as

$$N^{-1}\ln\Lambda=-2\pi P\mathrm{i}/LN+\int_{-\infty}^{\infty}\ln L(x,u)\delta_N(x)\,\mathrm{d}x \tag{21}$$

where $\delta_N(x)=N^{-1}\sum_{j=1}^{M}\delta(x-x_j)$. The function $\ln L(x,u)$ can be represented in the form

$$\ln L(x,u)=P\int_{-\infty}^{\infty}\frac{\sinh[(\pi-\lambda)q/2]}{q\sinh(\pi q/2)}\exp(uq+\mathrm{i}xq)\,\mathrm{d}q \tag{22}$$

where P here stands for the principal value. Using this representation and proceeding as in Hamer *et al* (1987), we find the intermediate expression for the finite-size correction as

$$\begin{aligned}\Delta(N^{-1}\ln\Lambda)&\equiv N^{-1}\ln\Lambda-\lim_{N\to\infty}(N^{-1}\ln\Lambda)\\&=-2\pi P\mathrm{i}/LN+\int_{-\infty}^{\infty}H(x)[\delta_N(x)-\sigma_N(x)]\,\mathrm{d}x\end{aligned} \tag{23}$$

where

$$\sigma_N(x)=\frac{\mathrm{d}}{\mathrm{d}x}Z_N(x)$$

and

$$H(x)=P\int_{-\infty}^{\infty}\frac{\exp(uq+\mathrm{i}qx)}{2q\cosh(\lambda q/2)}\,\mathrm{d}q. \tag{24}$$

The integral in (24) can be evaluated explicitly by residue calculus but it suffices here to note that, as $x\to\infty$,

$$H(x)\approx\mathrm{i}[\pi/2-2\cos(\pi u/\lambda)\exp(-\pi x/\lambda)]+2\sin(\pi u/\lambda)\exp(-\pi x/\lambda) \tag{25}$$

and $H(x)=H(-x)^*$. This is to be compared with equation (2.61) of Hamer *et al* (1987). Then, following Hamer *et al* all $1/N$ terms cancel out and after a straightforward calculation we finally obtain for large N

$$\Delta(N^{-1}\ln\Lambda)\approx-2\pi N^{-2}\sin(\pi u/\lambda)(X-c/12)-2\pi\mathrm{i}N^{-2}\cos(\pi u/\lambda)S \tag{26}$$

$$X=x_pQ^2+\frac{(\phi/2\pi)^2}{4x_p} \tag{27}$$

$$S=Q(\phi/2\pi) \tag{28}$$

where $c=1$ is the central charge, Q and $\phi/2\pi$ are given by (8) and (19) and

$$x_p=(\pi-\lambda)/2\pi=(L-s)/2L. \tag{29}$$

The sine and cosine factors in (26) are precisely the anisotropy factors discussed in Kim and Pearce (1987). The scaling dimensions X and the spins S agree with the expected operator content of the XXZ chain as discussed in Alcaraz *et al* (1988a, b). A more detailed analysis of the six-vertex model spectra appears in Karowski (1988).

Alcaraz *et al* (1988a) conjecture that the scaling dimensions given by (27), together with (19), are in fact the complete set for the primary operators in each charge Q sector. We also assume this. Following Alcaraz *et al* (1988a), we express the operator content in terms of the irreducible representations $(\Delta, \bar{\Delta})$ of the two commuting U(1) Kac–Moody algebras. If we denote by $E_P(\Sigma^l)$ the operator content of the csos model in the sector with Z_L charge P under the BC Σ^l, we then have

$$E_P(\Sigma^l) = \bigoplus_{m\in Z,n} \left(\frac{[2P/L+m+x_p(l+nL)]^2}{8x_p}, \frac{[2P/L+m-x_p(l+nL)]^2}{8x_p}\right) \tag{30}$$

where the sum over the winding number n is as given by (9).

4. Operator content for $g = \Sigma^l C$

We now consider the BC $g = \Sigma^l C$, $l \in S_L$. An allowed row configuration $(a_1, \boldsymbol{\sigma})$ should then satisfy

$$a_{N+1} = a_1 + \sum_{i=1}^{N} \sigma_i = l - a_1 \qquad (\text{mod } L). \tag{31}$$

When L is odd, there is always a unique height for a_1 that satisfies (31) for a given $\boldsymbol{\sigma}$. When L is even, there are always two such heights which differ by $L/2$. Therefore, there exists a one-to-one (two-to-one) correspondence when L is odd (even) between the csos states with $g = \Sigma^l$ and vertical arrow configurations.

First consider the L odd cases. Rows and columns of the row-to-row transfer matrix V_g can now be designated by $\boldsymbol{\sigma}$. Non-vanishing elements $V_g(\boldsymbol{\sigma}|\boldsymbol{\sigma}')$ of V_g are products of the six-vertex weights of (3). Let a_1 (a_1') be the height of the first column and M (M') be the total number of down arrows in a configuration $\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}'$). From (31) we have $a_1 - a_1' = M - M'$ (mod L). Also $|M - M'| \leq 1$ for a row of six-vertex configurations since the vertices of type 5 and 6 in figure 2 alternate. Therefore $V_g(\boldsymbol{\sigma}|\boldsymbol{\sigma}')$ is non-zero only when

$$|M - M'| = 1. \tag{32}$$

An extra down arrow in the top or bottom row of vertical bonds in a six-vertex configuration causes the first and last horizontal arrows to point in opposite directions. A typical configuration is shown in figure 3. Hence V_g is the same as the transfer

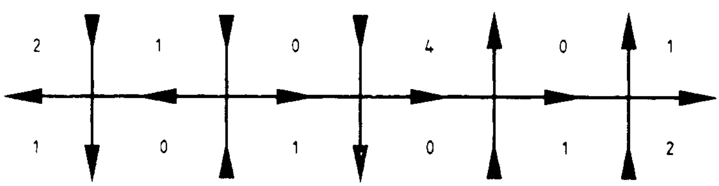

Figure 3. A typical vertex configuration associated to a non-vanishing element of the transfer matrix when $g = \Sigma^3 C$ for $L = 5$ and $N = 5$.

matrix of the six-vertex model under antiperiodic BC and is independent of l. Let $W_v(\mu, \sigma|\sigma', \nu)$ by the Boltzmann weight of a vertex where $\mu, \sigma, \sigma', \nu$ $(=\pm 1)$ designate the arrow directions of the four incident bonds as shown in figure 4. The sign convention is such that $+1$ indicates right or up arrows. For the weights given by (3), we have (Baxter 1982)

$$\begin{aligned} W_v(\mu, \sigma|\sigma', \nu) &= \tfrac{1}{2}\{[(1+W_5)+(1-W_5)\mu\sigma]\delta(\mu, \sigma')\delta(\sigma, \nu) \\ &\quad + W_3(1-\mu\sigma)\delta(\mu, -\sigma')\delta(\sigma, -\nu)\} \end{aligned} \tag{33}$$

where $\delta(a, b)$ is the Kronecker delta function. Using (33), we can write $V_g(\boldsymbol{\sigma}|\boldsymbol{\sigma}')$ as

$$V_g(\boldsymbol{\sigma}|\boldsymbol{\sigma}') = \Sigma \prod_{i=1}^{N} W_v(\mu_i, \sigma_i|\sigma'_i, \mu_{i+1}) \tag{34}$$

where $\mu_{N+1} = -\mu_1$ and the sum is over the horizontal configurations $\{\mu_1, \ldots, \mu_N\}$.

The transfer matrix (34) has an arrow reversal quantum number. In terms of the CSOS language it is the quantum number ± 1 of the operator $\Sigma' C$. On the other hand, the charge Q is not conserved due to (32) and the Bethe ansatz cannot be applied to (34). However, the spectral parameter u does not play a role in the operator content except through the anisotropy scaling factor as indicated in (26) (Kim and Pearce 1987). Therefore we may take the logarithmic derivative of (34) with respect to u at $u=0$, thereby mapping the model to the XXZ chain. Following the standard procedure (Baxter 1982, p 260), we find, as $u \to 0$,

$$V_g \approx V_g^0 \exp[-\sin(\pi u/\lambda)\hat{H}] \tag{35}$$

where V_g^0 is (34) at $u=0$ and $\hat{H}$ is given by

$$\hat{H} = -\frac{\lambda}{2\pi \sin\lambda} \sum_{i=1}^{N} [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \cos\lambda(\sigma_i^z \sigma_{i+1}^z - 1)] \tag{36}$$

with the BC

$$\sigma_{N+1}^x = \sigma_1^x \qquad \sigma_{N+1}^y = -\sigma_1^y \qquad \sigma_{N+1}^z = -\sigma_1^z. \tag{37}$$

Here σ^x, σ^y and σ^z are the Pauli spin operators. The arrow reversal operation in (34) is translated into the charge conjugation operator $\hat{C}$ in the XXZ model and (37) is the charge conjugation BC.

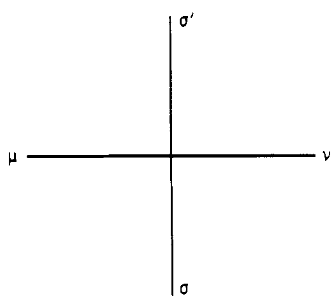

Figure 4. The index convention for the vertex weight.

Let us denote the operator content of the csos model under the BC $\Sigma^l C$ in the sector $\Sigma^l C = \nu$ $(=\pm 1)$ by $E_\nu(\Sigma^l)$. The operator content is then the same as that of (36) and (37) in the sector $\hat{C} = \nu$. In (36) and (37), there is an extra quantum number associated with the parity of the number of down arrows for even N. However, this is not a quantum number for (34). Using the result of Alcaraz *et al* (1988a), we obtain

$$\begin{aligned} E_+(\Sigma^l C) &= 2(\{\tfrac{1}{16}\}, \{\tfrac{1}{16}\}) \oplus 2(\{\tfrac{9}{16}\}, \{\tfrac{9}{16}\}) \\ E_-(\Sigma^l C) &= 2(\{\tfrac{1}{16}\}, \{\tfrac{9}{16}\}) \oplus 2(\{\tfrac{9}{16}\}, \{\tfrac{1}{16}\}) \end{aligned} \tag{38}$$

for N even and

$$E_+(\Sigma^l C) = E_-(\Sigma^l C) = (\{\tfrac{1}{16}\} \oplus \{\tfrac{9}{16}\}, \{\tfrac{1}{16}\} \oplus \{\tfrac{9}{16}\}) \tag{39}$$

for N odd. Here

$$\begin{aligned} \{\tfrac{1}{16}\} &= \bigoplus_{m \in Z} \left(\frac{(1+8m)^2}{16} \right) \\ \{\tfrac{9}{16}\} &= \bigoplus_{m \in Z} \left(\frac{(3+8m)^2}{16} \right) \end{aligned} \tag{40}$$

as defined in Alcaraz *et al* (1988a).

We next consider the L even cases. For a given $\boldsymbol{\sigma}$, there exist two states connected by $\Sigma^{L/2}$; $(a, \boldsymbol{\sigma})$ and $(a + L/2, \boldsymbol{\sigma})$ where $a = l/2 - Q$. For any non-vanishing matrix element $V_g(a, \boldsymbol{\sigma} \mid a', \boldsymbol{\sigma}')$ $(|a - a'| = 1)$, we have

$$\begin{aligned} V_g(a, \boldsymbol{\sigma} \mid a', \boldsymbol{\sigma}') &= V_g(a + L/2, \boldsymbol{\sigma} \mid a' + L/2, \boldsymbol{\sigma}') \\ V_g(a, \boldsymbol{\sigma} \mid a' + L/2, \boldsymbol{\sigma}') &= V_g(a + L/2, \boldsymbol{\sigma} \mid a', \boldsymbol{\sigma}') = 0. \end{aligned} \tag{41}$$

Therefore we can separate $\boldsymbol{V}_g$ into two identical diagonal sectors, each of which is itself the same as (34). We designate these two sectors with the quantum number of $\Sigma^{L/2}$. When L is even, we have $N + l$ even. Consequently the operator content for each $\Sigma^{L/2} = \pm 1$ sector is given by (38) when l is even and by (39) when l is odd. Denoting the operator content for the sector $\Sigma^{L/2} = \nu$ and $\Sigma^l C = \nu'$ by $E_{\nu,\nu'}(\Sigma^l C)$, we have

$$\begin{aligned} E_{++}(\Sigma^l C) = E_{-+}(\Sigma^l C) &= 2(\{\tfrac{1}{16}\}, \{\tfrac{1}{16}\}) \oplus 2(\{\tfrac{9}{16}\}, \{\tfrac{9}{16}\}) \\ E_{+-}(\Sigma^l C) = E_{--}(\Sigma^l C) &= 2(\{\tfrac{1}{16}\}, \{\tfrac{9}{16}\}) \oplus 2(\{\tfrac{9}{16}\}, \{\tfrac{1}{16}\}) \end{aligned} \tag{42}$$

for l even and

$$E_{++}(\Sigma^l C) = E_{-+}(\Sigma^l C) = E_{+-}(\Sigma^l C) = E_{--}(\Sigma^l C) = (\{\tfrac{1}{16}\} \oplus \{\tfrac{9}{16}\}, \{\tfrac{1}{16}\} \oplus \{\tfrac{9}{16}\}) \tag{43}$$

for l odd.

5. Discussion

We have obtained the full operator content of the critical csos transfer matrix under general BC. This is done by mapping the csos model to the six-vertex model and the XXZ chain. The completeness of our results is, in fact, a conjecture based on the XXZ chain results of Alcaraz *et al* (1988a). As demanded by consistency, the operator content for g belonging to the same conjugacy class is the same. Moreover, although formula (30) applies for $g = \Sigma^l$ and general l, the extra symmetry for Σ^0, and also for

$\Sigma^{L/2}$ when L is even, is reflected in the appearance of extra degeneracies in (30) when $l=0$ or $L/2$.

The scaling dimensions appearing in (30) are of the form

$$X=\frac{L-s}{8L}(l+nL)^2+\frac{(2P+mL)^2}{2L(L-s)} \tag{44}$$

where (29) is used. Pearce and Seaton (1989) obtained the critical exponents for $2[(L-1)/2]$ complex order parameters, $[\cdots]$ denoting the integer part. The scaling dimensions corresponding to these order parameters are found to be

$$\frac{(2j)^2}{2L(L-s)} \qquad \frac{(L-2j)^2}{2L(L-s)} \qquad j=1,2,\ldots,[(L-1)/2] \tag{45}$$

each repeated twice. These exponents are generated in the second term on the right-hand side of (44) by choosing $P=j$, $m=0$ or -1 and $P=L-j$, $m=-2$ or -1. The exponent associated with the temperature-like variable, the nome of elliptic functions, is $2(L-s)/L$ and is also contained in the second term in (44). When $s>L/2$ the temperature-like variable also plays the role of a field conjugate to one of the order parameters.

The set

$$\{2P+mL\,|\,P\in S_L,\ m\in \boldsymbol{Z}\} \tag{46}$$

appearing in (30) and (44) is equal to $\boldsymbol{Z}$ for L odd but is $2\boldsymbol{Z}\oplus 2\boldsymbol{Z}$ for L even. Therefore each operator appears exactly twice for L even. This is due to the fact that, for L even and hence $N+l$ even, the lattice is decomposed into two sublattices, each having all even or all odd heights and, the two possibilities being equivalent, the partition function picks up a trivial factor of 2 (Andrews *et al* 1984). This does not contradict the Perron–Frobenius theorem. One can remove this trivial degeneracy either by taking m to be even integers or by restricting P to take the values $0,1,\ldots,L/2-1$. When $L/2$ is odd, the same effect is achieved by restricting P to be even or odd.

von Gehlen *et al* (1988) have studied L-state ($L\geqslant 5$) quantum chains having the same D_L symmetry. When L is even, the operator content they obtain in the $c=1$ region of the models is the same as our result for all BC as long as all sectors are combined and the degeneracy discussed above is taken into account. In this correspondence, the coupling constant g in equation (5.7) of von Gehlen *et al* (1988) is identified as $x_pL/2=(L-s)/4$. However, the distribution of operators into various sectors is slightly different when the BC is $\Sigma^l C$. It depends on whether l is even or odd in the CSOS model as given by (42) and (43), whereas it is given by (42), apart from the factor of 2, for all l in the quantum chain. Pasquier (1987c) constructed a modular invariant partition function for a CSOS model on a torus under periodic boundary conditions. Our result (30) for $g=\Sigma^0$ and for L even extends his result. In particular, the coupling constant $g=1$ in equation (18) of Pasquier (1987c) is replaced by $g=(L-s)/L$.

When L is odd, the operator content of the CSOS model depends on the parity of N. This is a characteristic of models involving sublattice symmetry breaking. For the interacting hard square model (Baxter and Pearce 1983) and the magnetic hard square model (Pearce and Kim 1987), the operator content under periodic boundary conditions with odd N is given by that of the corresponding quantum models with antiperiodic (Kim 1988) and the twisted (Kim *et al* 1988) BC, respectively. This is naturally explained by considering extended models where the spin states now carry the sublattice indices (Choi *et al* 1989). It is in this sense that the four-state RSOS A_4 model of Andrews *et* *al* and the six-state $D_5^{(1)}$ (Kuniba and Yajima 1988b) or $\hat{D}_5$ (Pasquier 1987a, b) model are equivalent to the hard square and the magnetic hard square model, respectively. Thus, to compare our odd L result with the quantum chains, we are led to consider extended models when L is odd. In the extended models, the heights can take $2L$ values $\{0, 1, \ldots, L-1, \bar{0}, \bar{1}, \ldots, \overline{L-1}\}$ and the adjacency condition is as shown in figure 5. The face weights remain the same as in the L-state model and do not distinguish between the states a and $\bar{a}$. Simple relabelling of the heights then reduces it to the $2L$-state CSOS model. In the extended scheme, $\Sigma^{2l'}$ and $\Sigma^{2l'}C$, $l' \in S_L$, are the BC which do not mix the two sublattices. The transfer matrix of the extended model under the BC Σ^l and Σ^{l+L}, $l \in S_L$, can both be put in the block form

$$\begin{pmatrix} 0 & V_g \\ V_g & 0 \end{pmatrix} \tag{47}$$

where V_g is the transfer matrix of the original L-state model under the same BC $g = \Sigma^l$ but with $N + l$ even (odd) for the former (latter). The same is also true for $\Sigma^l C$ and $\Sigma^{l+L}C$. Using this connection, one can, in fact, derive the odd L result from the even $2L$ result. If we consider only the subset of the BC $\Sigma^{2l'}$ and $\Sigma^{2l'}C$, $l' \in S_L$, in the extended model, the operator content is then the same, sector by sector, as that of the L-state (L odd) quantum model under the BC $\Sigma^{l'}$ and $\Sigma^{l'}C$, respectively. In this case the coupling constant of von Gehlen *et al* (1988) is identified as $2Lx_p = L - s$.

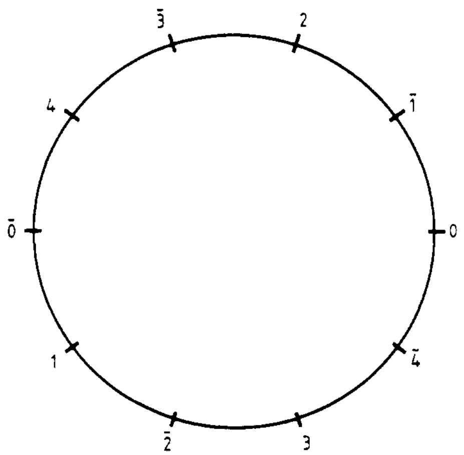

Figure 5. The adjacency diagram for the extended $2L$-state model for $L = 5$.

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